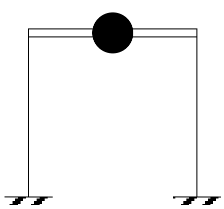
Multi-Degree-Of-Freedom (MDOF) Systems and Modal Analysis

Ahmed Elgamal

SDOF Shear Building (rigid roof)



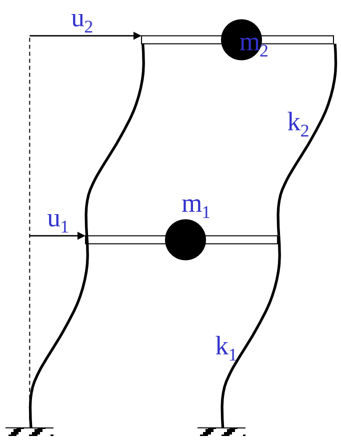
$$m\ddot{u} + ku + c\dot{u} = -m\ddot{u}_g$$

m = lumped mass = $m_{roof} + 2\,(1/2\ m_{col})$

$$k = 2k_{col} = 2\frac{12EI_c}{h^3} = \frac{24EI_c}{h^3}$$

Ahmed Elgamal

2-Story Shear Building (2-DOF system)



$$m_2 = m_2\text{roof} + 2\left(\frac{1}{2} m_{col}\right)$$

$$m_1 = m_1\text{floor} + 4\left(\frac{1}{2} m_{col}\right)$$

$$k_1 = 2k_{col1} \qquad k_2 = 2k_{col2}$$

$$m_2\ddot{u}_2 + k_2(u_2 - u_1) = -m_2\ddot{u}_g$$

$$m_1\ddot{u}_1 + k_2(u_1 - u_2) + k_1u_1 = -m_1\ddot{u}_g$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}\ddot{u}_g$$

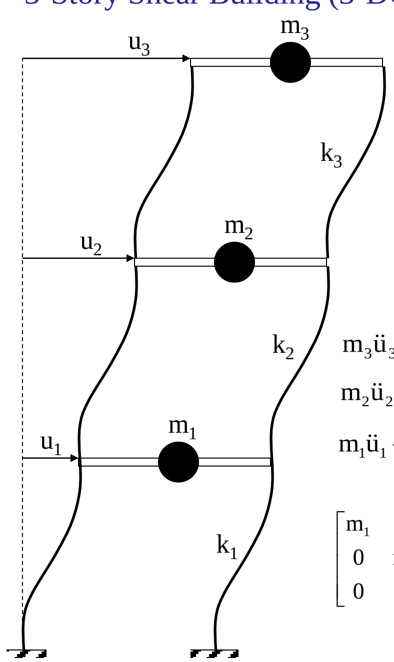
or, $\underline{m}\underline{\ddot{u}} + \underline{k}\underline{u} = -\underline{m}\underline{1}\ddot{u}_g$

$$\mathbf{m\ddot{u}} + \mathbf{ku} = -\mathbf{m1}\ddot{u}_g$$

where $\underline{1}$ or **1** is the Identity Matrix

Ahmed Elgamal

3-Story Shear Building (3-DOF system)



$$m_3\ddot{u}_3 + k_3(u_3 - u_2) = -m_3\ddot{u}_g$$

$$m_2\ddot{u}_2 + k_3(u_2 - u_3) + k_2(u_2 - u_1) = -m_2\ddot{u}_g$$

$$m_1\ddot{u}_1 + k_2(u_1 - u_2) + k_1u_1 = -m_1\ddot{u}_g$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = -\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}\ddot{u}_g$$

For a N-DOF system,

Ahmed Elgamal

$$\mathbf{m} = \begin{bmatrix} m_1 &&&&&&&&& \\ & m_2 &&&&&&&& \\ && - &&&&&&& \\ &&& - &&&&&& \\ &&&& - &&&&& \\ &&&&& - &&&& \\ &&&&&& m_m &&& \\ &&&&&&& - && \\ &&&&&&&& - & \\ &&&&&&&&& m_n \end{bmatrix}$$

$$\mathbf{k} = \begin{bmatrix} (k_1+k_2) & -k_2 & 0 & - & - & - & - & - & - & 0 \\ -k_2 & (k_2+k_3) & -k_3 & 0 & - & - & - & - & - & 0 \\ 0 && - &&&&&&& \\ 0 &&& - &&&&&& \\ | &&&& - &&&&& \\ | &&&&& - &&&& \\ 0 &&&& 0 & -k_m & (k_m+k_{m+1}) & -k_{m+1} & 0 & \\ | &&&&&&& - && 0 \\ | &&&&&&&& - & -k_n \\ 0 & 0 & - & - & - & - & - & 0 & -k_n & k_n \end{bmatrix}$$

Equation of motion for a N-DOF system,

Ahmed Elgamal

$$\begin{bmatrix} m_1 &&&&&&&&& \\ & m_2 &&&&&&&& \\ && - &&&&&&& \\ &&& - &&&&&& \\ &&&& - &&&&& \\ &&&&& - &&&& \\ &&&&&& m_k &&& \\ &&&&&&& - && \\ &&&&&&&& - & \\ &&&&&&&&& m_n \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ | \\ | \\ | \\ | \\ \ddot{u}_m \\ | \\ | \\ \ddot{u}_n \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 & 0 & - & - & - & - & - & - & 0 \\ -k_2 & (k_2+k_3) & -k_3 & 0 & - & - & - & - & - & 0 \\ 0 && - &&&&&&& \\ 0 &&& - &&&&&& \\ | &&&& - &&&&& \\ | &&&&& - &&&& \\ 0 &&&& 0 & -k_m & (k_m+k_{m+1}) & -k_{m+1} & 0 & \\ | &&&&&&& - && 0 \\ | &&&&&&&& - & -k_n \\ 0 & 0 & - & - & - & - & - & 0 & -k_n & k_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ | \\ | \\ | \\ | \\ u_m \\ | \\ | \\ u_n \end{bmatrix} = - \begin{bmatrix} m_1 \\ m_2 \\ | \\ | \\ | \\ | \\ m_m \\ | \\ | \\ m_n \end{bmatrix} \ddot{u}_g$$

or (with damping included):

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m1}\ddot{u}_g$$

with initial conditions: $\mathbf{u} = \mathbf{u}(t=0)$

$\dot{\mathbf{u}} = \dot{\mathbf{u}}(t=0)$

Natural Frequencies of a N-DOF system

Similar to the SDOF system, MDOF systems have natural frequencies. A *2*-DOF has 2 natural frequencies ω_1 and ω_2, and a *n*-DOF system has natural frequencies $\omega_1, \omega_2, \ldots, \omega_n$

Similar to the SDOF, free vibration involves the system response in its natural frequencies. The corresponding Free Vibration Equation is (with no damping):

$$\mathbf{m\ddot{u} + ku = 0}$$

In free vibration, the system will oscillate in a steady-state harmonic fashion, such that:

$$\mathbf{\ddot{u}} = -\omega^2 \mathbf{u}$$

e.g. $u = a \cdot \sin(\omega t) + b \cdot \cos(\omega t)$ gives $\ddot{u} = -\omega^2 u$

substituting for $\mathbf{\ddot{u}}$, we get:

$$\left(-\omega^2 \mathbf{m} + \mathbf{k}\right)\mathbf{u} = \mathbf{0}$$

or

$$\left(\mathbf{k} - \omega^2 \mathbf{m}\right)\mathbf{u} = \mathbf{0} \qquad \text{Equation 1}$$

The above equation represents a ***classic*** problem in Math/Physics, known as the ***Eigen-value*** problem.

The ***trivial*** solution of this problem is **u** = **0** (i.e., nothing is happening, and the system is at rest).

For a non-trivial solution (which will allow for computing the natural frequencies during free vibration), the determinant of $(\mathbf{k} - \omega^2\mathbf{m})$ must be equal to zero such that:

$|\mathbf{k} - \omega^2\mathbf{m}| = \mathbf{0}$ or $|\mathbf{k} - \lambda\mathbf{m}| = \mathbf{0}$ where $\lambda = \omega^2$

For a 2-DOF system for instance (see next page), the above determinant calculation will result in a quadratic equation in the unknown term λ . If this quadratic equation is solved (by hand), two roots are found (λ_1 and λ_2), which define ω_1 and ω_2 (the natural resonant frequencies of this 2-DOF system).

9

Ahmed Elgamal

2-Story Shear Building (2-DOF system)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \ddot{u}_g$$

$$(\mathbf{k} - \lambda\mathbf{m})\mathbf{u} = \mathbf{0}$$

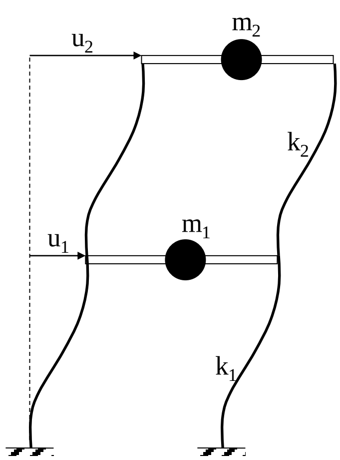
$$\begin{bmatrix} (k_1 + k_2) - \lambda m_1 & -k_2 \\ -k_2 & k_2 - \lambda m_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Set Determinant = 0:

$$\begin{vmatrix} (k_1 + k_2) - \lambda m_1 & -k_2 \\ -k_2 & k_2 - \lambda m_2 \end{vmatrix} = 0$$

or, $((k_1 + k_2) - \lambda m_1)(k_2 - \lambda m_2) - (-k_2)(-k_2) = 0$

$$(m_1 m_2)\lambda^2 - ((m_1 k_2 + m_2(k_1 + k_2))\lambda + (k_1 k_2) = 0$$



Solve for the λ_1 and λ_2 using the standard approach

$$a\lambda^2 + b\lambda + c = 0 \qquad \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: For $\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$

Determinant = $WZ - XY$

10

For a general N-DOF system:

Matlab or similar computer program can be used to solve the determinant equation (of order equal to the NDOF system), defining NDOF roots or NDOF natural frequencies

$\omega_1 , \omega_2 ,\ldots, \omega_{NDOF}$

Note:
These resonant (natural) frequencies ω_1 , ω_2 , … are conventionally ordered lowest to highest (e.g., $\omega_1 = 8$ radians, $\omega_2 = 14$ radians, and so forth).

Mode Shapes

Steady State vibration at any of the resonant frequencies ω_n takes place in the form of a special oscillatory shape, know as the corresponding **mode shape** $\boldsymbol{\phi_n}$

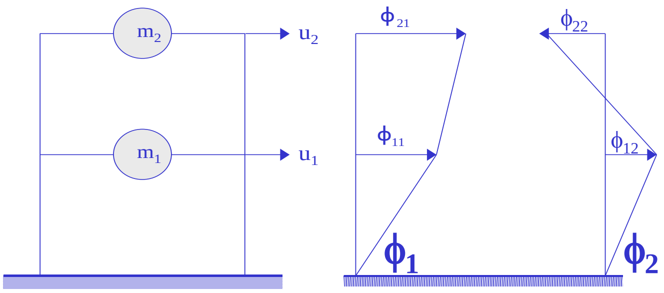
To define these mode shapes (one for each identified ω_n), go ahead and substitute the value of ω_n for ω in Eq. 1

$$(\mathbf{k} - \omega^2 \mathbf{m})\mathbf{u} = \mathbf{0}$$

and solve for the vector **u** which will define the corresponding mode shape $\boldsymbol{\phi_n}$:

$$(\mathbf{k} - \omega_n^2 \mathbf{m})\boldsymbol{\phi}_n = \mathbf{0}$$

2-DOF system (2 mode shapes ϕ_1 and ϕ_2)



$$\phi_1 = \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix}$$

Note: Any mode shape ϕ_n only defines ***relative*** amplitudes of motion of the different degrees of freedom in the MDOF system. For instance, if you are solving a 2-DOF system, you might end up with something like (when solving for the first mode):

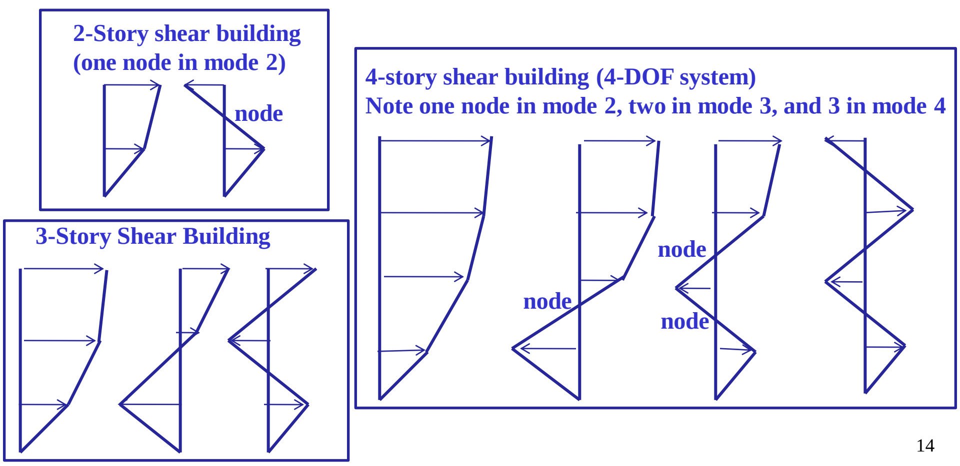
$\phi_{11} - 2\phi_{21} = 0$, only defining a ratio between amplitudes of ϕ_{11} and ϕ_{21}

(for instance, if $\phi_{11} = 1$, then $\phi_{21} = 0.5$, or if you choose $\phi_{11} = 2$, then $\phi_{21} = 1$, and so forth).

Generally, go ahead and make $\phi_{mn} = 1$ (where m is top floor Dof and n is mode shape number) and solve for the other degrees of freedom in the vector ϕ_n

Note: When you substitute any of the ω_n values into Eq. 1, the determinant of the matrix **(k-ω_n^2m)** automatically becomes = 0, since this ω_n is a root of the determinant equation (i.e., the matrix becomes singular).
The determinant being zero is a necessary condition for obtaining a vector **u** (the mode shape ϕ_n) that is not equal to zero (i.e., a solution other than the trivial solution of **u** = **0**.

Sample Mode shape Configurations



Properties of ϕ_n

a) Mode shapes are orthogonal such that (for any n≠r)

$$\boldsymbol{\phi}_n^T \mathbf{k} \boldsymbol{\phi}_r = \boldsymbol{\phi}_n^T \mathbf{m} \boldsymbol{\phi}_r = 0 \qquad (\text{not } \boldsymbol{\phi}_n^T \boldsymbol{\phi}_r = 0)$$

b) For any mode $\boldsymbol{\phi}_n$, modal mass M_n is defined by:

$$\boldsymbol{\phi}_n^T \mathbf{m} \boldsymbol{\phi}_n = M_n$$

c) For any mode $\boldsymbol{\phi}_n$, modal stiffness K_n is defined by:

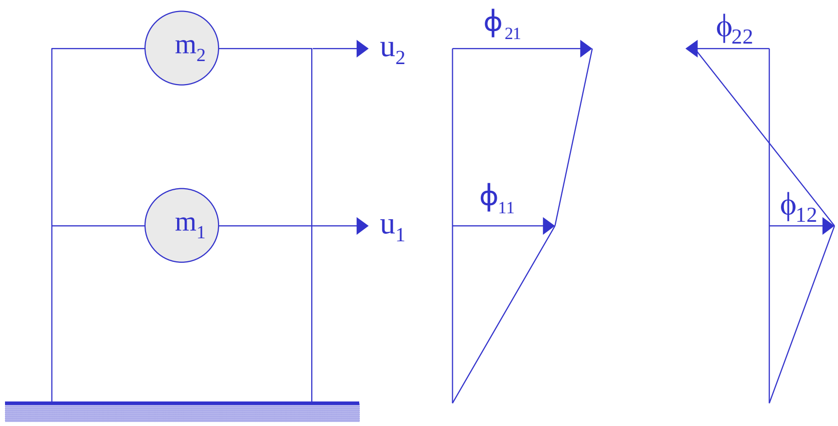
$$\boldsymbol{\phi}_n^T \mathbf{k} \boldsymbol{\phi}_n = K_n = \omega_n^2 M_n$$

d) If $\boldsymbol{\phi}_n^T \mathbf{m} \boldsymbol{\phi}_n = 1.0$ then $\boldsymbol{\phi}_n^T \mathbf{k} \boldsymbol{\phi}_n = \omega_n^2$

To do that, multiply each component of mode $\boldsymbol{\phi}_n$ by $\dfrac{1}{\sqrt{M_n}}$

Solution of by Mode superposition

Example of a 2-DOF system (2 mode shapes $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$)



Modal Analysis (Solution of MDOF equation of motion by Mode Superposition)

The solution u will be represented by a summation of the mode shapes ϕ_n, each multiplied by a scaling factor q_n (known as the generalized coordinate) . For instance, for the 2-DOF system:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} q_1 + \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} q_2 = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2]\mathbf{q} = \boldsymbol{\Phi}\mathbf{q}$$

In the above, $\boldsymbol{\Phi}$ is known as the modal matrix. As such, changes in the displaced shape of the structure **u** with time will be captured by the time histories of the vector **q**

Substituting $\mathbf{u} = \boldsymbol{\Phi}\mathbf{q}$ in the equation of motion $\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\mathbf{1}\ddot{u}_g$

Results in $\mathbf{m}\boldsymbol{\Phi}\ddot{\mathbf{q}} + \mathbf{k}\boldsymbol{\Phi}\mathbf{q} = -\mathbf{m}\mathbf{1}\ddot{u}_g$

To benefit from the mode orthogonality property, multiply by $\boldsymbol{\Phi}^T$ to get:

$$\boldsymbol{\Phi}^T\mathbf{m}\boldsymbol{\Phi}\ddot{\mathbf{q}} + \boldsymbol{\Phi}^T\mathbf{k}\boldsymbol{\Phi}\mathbf{q} = -\boldsymbol{\Phi}^T\mathbf{m}\mathbf{1}\ddot{u}_g$$

or

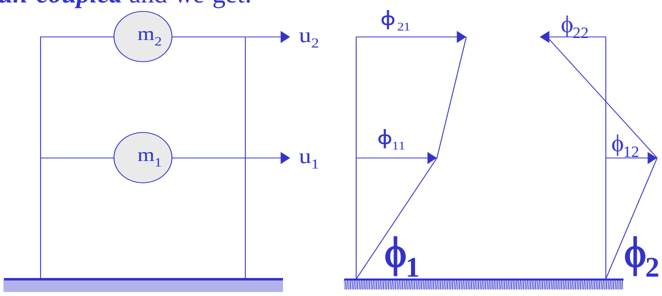
$$\begin{bmatrix} \boldsymbol{\phi}_1^T\mathbf{m}\boldsymbol{\phi}_1 & \boldsymbol{\phi}_1^T\mathbf{m}\boldsymbol{\phi}_2 \\ \boldsymbol{\phi}_2^T\mathbf{m}\boldsymbol{\phi}_1 & \boldsymbol{\phi}_2^T\mathbf{m}\boldsymbol{\phi}_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{\phi}_1^T\mathbf{k}\boldsymbol{\phi}_1 & \boldsymbol{\phi}_1^T\mathbf{k}\boldsymbol{\phi}_2 \\ \boldsymbol{\phi}_2^T\mathbf{k}\boldsymbol{\phi}_1 & \boldsymbol{\phi}_2^T\mathbf{k}\boldsymbol{\phi}_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = -\begin{bmatrix} \boldsymbol{\phi}_1^T\mathbf{m}\mathbf{1} \\ \boldsymbol{\phi}_2^T\mathbf{m}\mathbf{1} \end{bmatrix} \ddot{u}_g$$

Due to the ***orthogonality*** property of mode shapes (see previous slide), the matrix equation becomes ***un-coupled*** and we get:

$$M_1\ddot{q}_1 + M_1\omega_1^2 q_1 = -L_1\ddot{u}_g$$
$$M_2\ddot{q}_2 + M_2\omega_2^2 q_2 = -L_2\ddot{u}_g$$

or,

$$\ddot{q}_1 + \omega_1^2 q_1 = -\frac{L_1}{M_1}\ddot{u}_g$$
$$\ddot{q}_2 + \omega_2^2 q_2 = -\frac{L_2}{M_2}\ddot{u}_g$$



For a diagonal mass matrix: $M_i = \sum_{j=1}^{NDOF} m_j\phi_{ji}^2 \qquad L_i = \sum_{j=1}^{NDOF} m_j\phi_{ji}$

The terms L_1/M_1 and L_2/M_2 are known as modal participation factors. These terms control the influence of $\ddot{u}_g$ on the modal response. You may notice that (if both modes are normalized to 1.0 at roof level for example) $L_1/M_1 > L_2/M_2$ since ϕ_{11} and ϕ_{21} are of the same sign while ϕ_{12} and ϕ_{22} are of opposite signs. Therefore, the first mode is likely to play a more prominent role in the overall response (frequency content of the input ground motion also affects this issue).

Note that the original coupled matrix Eq. of motion, has now become a set of ***un-coupled*** equations. You can solve each one separately (as a SDOF system), and compute histories of q_1 and q_2 and their time derivatives. To compute the system response, plug the **q** vector back into Equation 2 and get the **u** vector

$$\mathbf{u} = \mathbf{\Phi q}$$

(the same for the time derivatives to get relative velocity and acceleration).

The beauty here is that there is no matrix operations involved, since the matrix equation of motion has become a set of un-coupled equation, each including only one generalized coordinate q_n.

Damping in a Modal Solution

Now, you can add any modal damping you wish (which is another big plus, since you control the damping in each mode individually). If you choose ξ_i = 0.02 or 0.05, the equations become:

$$\ddot{q}_i + 2\xi_i\omega_i\dot{q}_i + \omega_i^2 q_i = -\frac{L_i}{M_i}\ddot{u}_g \;, \quad i = 1, 2, \ldots \text{NDOF}$$

OK, go ahead now and solve for $q_i(t)$ in the above uncoupled equations (using a SDOF-type program), and the final solution is obtained from:

$$\mathbf{u} = \mathbf{\Phi q}$$
$$\dot{\mathbf{u}} = \mathbf{\Phi}\dot{\mathbf{q}}$$
$$\ddot{\mathbf{u}} = \mathbf{\Phi}\ddot{\mathbf{q}}$$
$$\ddot{\mathbf{u}}^t = \ddot{\mathbf{u}} + \mathbf{1}\ddot{u}_g$$

Modal Analysis (3-DOF system)

The solution u will be represented by a summation of the mode shapes ϕ_n, each multiplied by a scaling factor q_n (known as the generalized coordinate) . For instance, for the 3-DOF system:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{bmatrix} q_1 + \begin{bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{bmatrix} q_2 + \begin{bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{bmatrix} q_3 = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \boldsymbol{\phi}_3][\mathbf{q}] = \boldsymbol{\Phi}\mathbf{q}$$

In the above, $\boldsymbol{\Phi}$ is known as the modal matrix. As such, changes in the displaced shape of the structure $\mathbf{u}$ with time will be captured by the time histories of the vector $\mathbf{q}$
Note: If a two mode solution is sought, the system above becomes:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{bmatrix} q_1 + \begin{bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{bmatrix} q_2 = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \\ \phi_{31} & \phi_{32} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2][\mathbf{q}] = \boldsymbol{\Phi}\mathbf{q}$$

Note: If a single (1^{st} or fundamental) mode solution is sought, the system above becomes:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{bmatrix} q_1 = [\boldsymbol{\phi}_1] q_1$$

Multi-Degree-Of-Freedom (MDOF) Response Spectrum Procedure

1. Once you have generalized coordinates and uncoupled equations, use response spectrum to get maximum values of response $(r_i)_{max}$ for each mode separately.

Calculate expected max response ($\bar{r}$) using $\bar{r}_{max} = \sqrt{\sum (r_i)^2_{max}}$ ← **root sum square formula**

where i = 1, 2, … N degrees of freedom of interest (maybe first 4 modes at most) and r is any quantity of interest such as $|u_{max}|$ or SD

(note that summing the maxima from each mode directly is typically too conservative and is therefore not popular; because the maxima occur at different time instants during the earthquake excitation phase)

See A. Chopra "Dynamics of Structures" for improved formulae to estimate $\bar{r}_{max}$.

Response Spectrum Modal Responses

Max relative displacement $|\mathbf{u}_n|$ or $|u_{jn}|$ (j^{th} floor, n^{th} mode)

$$u_{jn} = \frac{L_n}{M_n} S_{dn} \phi_{jn}$$

(S_{dn} is S_d evaluated at frequency ω_n or period T_n)

Estimate of maximum floor displacement

$$|u_j| = \sqrt{\sum_{n=1}^{M} u_{jn}^2}$$

(M = number of modes of interest)

Maximum Equivalent static force $\mathbf{f}_n$ or f_{jn} (j^{th} floor, n^{th} mode)

$$f_{jn} = \frac{L_n}{M_n} S_{an} m_j \phi_{jn}$$

Therefore, modal base shear V_{0n} and moment M_{0n}

of floors

$$V_{0n} = \sum_{j=1}^{N} f_{jn}$$

base

$$M_{0n} = \sum_{j=1}^{N} f_{jn} d_j$$

where d_j = Distance from floor j to base

Estimate of maximum base shear and moment:

$$|V_0| = \sqrt{\sum_{n=1}^{M} V_{0n}^2}$$

$$|M_0| = \sqrt{\sum_{n=1}^{M} M_{0n}^2}$$

Damping Matrix for MDOF Systems

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\mathbf{1}\ddot{u}_g$$

Mass-proportional damping

$\mathbf{c} = a_o \mathbf{m}$

Stiffness-proportional damping

$\mathbf{c} = a_1 \mathbf{k}$

Classical damping (Rayleigh damping)

$$\mathbf{c} = a_0 \mathbf{m} + a_1 \mathbf{k}$$

Stiffness proportional damping appeals to intuition because it generates damping based on story deformations. However, mass proportional damping may be needed as will be shown below.

Mass-proportional damping: c = a_o **m**

Defining a_0 to obtain a desired modal damping ζ_n in mode n

In any modal equation, we have

$$M_n \ddot{q}_n + C_n \dot{q}_n + K_n q_n = 0$$

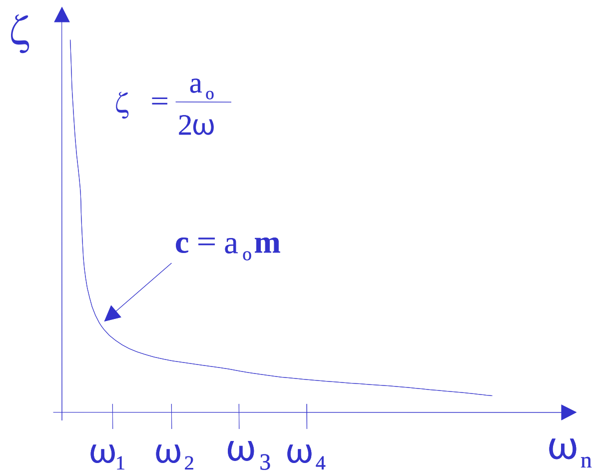
where, $C_n = 2\zeta_n \omega_n M_n$

Therefore, a_o can be specified to obtain any desired ζ_n for a given mode n such that $C_n = a_0 M_n$

$2\zeta_n \omega_n M_n = a_0 M_n$ or $a_0 = 2\zeta_n \omega_n$

(e.g. at $\omega_1 = 2\pi$ radians/s, $\zeta_1 = .05$) → find a_0

With a_0 defined by $a_0 = 2\,\zeta_n\,\omega_n$, this form of mass proportional damping will change with frequency according to $\boldsymbol{\zeta = a_0 / 2\omega}$ as shown in the figure below.



Stiffness-proportional damping: c = a_1 **k**

Defining a_1 to obtain a desired modal damping ζ_n

In any modal equation, we have

$$M_n\ddot{q}_n + C_n\dot{q}_n + K_nq_n = 0$$

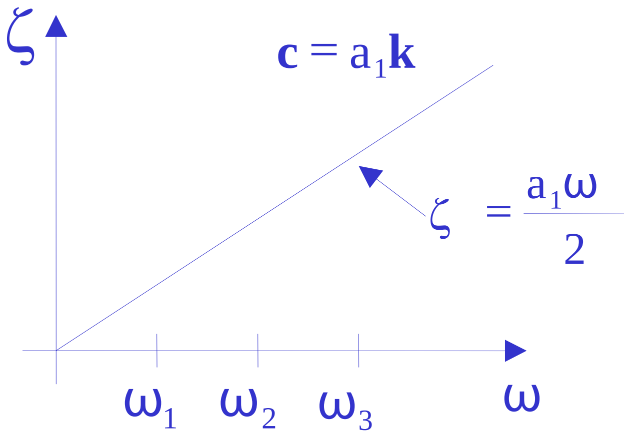
where, $C_n = 2\zeta_n\omega_nM_n$ and $K_n = \omega_n^2M_n$

Therefore, a_o can be specified to obtain any desired ξ_n for a given mode n such that $C_n = a_1K_n$, or:

$$2\zeta_n\omega_nM_n = a_1\omega_n^2M_n \quad \text{or} \quad a_1 = 2\zeta_n/\omega_n$$

(e.g. at $\omega_1 = 2\pi$ radians/s, $\zeta_1 = .05$) → find a_1

With a_1 defined by $a_1 = 2\,\zeta_n / \omega_n$, this form of stiffness proportional damping will change with frequency according to $\boldsymbol{\zeta = a_1\omega / 2}$ as shown in the figure below (damping increases linearly with frequency.



Physically, we often observe (in first approximation) a nearly equal value of damping for the first few modes of structural response (e.g., first 1- 4 modes or so), and we want to model that. Therefore, we use (Classical or Rayleigh damping):

$$\mathbf{c} = a_0\mathbf{m} + a_1\mathbf{k}$$

$$2\zeta_n\omega_n M_n = a_0 M_n + a_1\omega_n^2 M_n$$

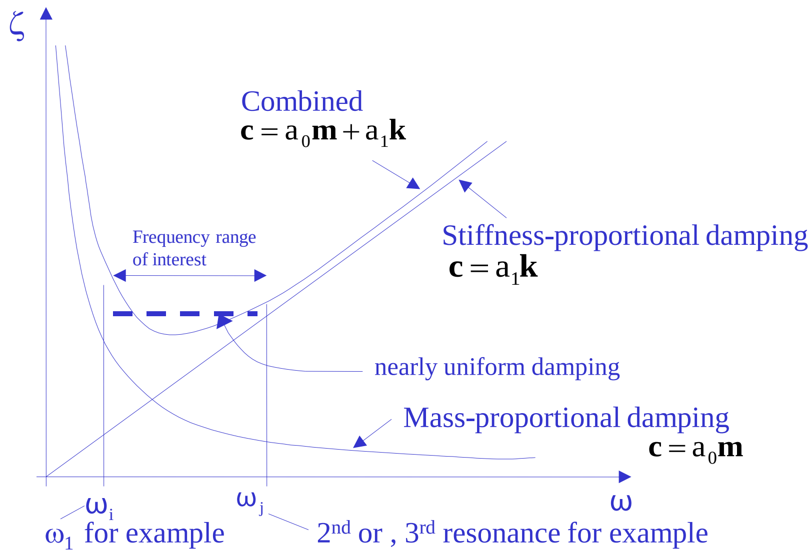
$$\zeta_n = (a_0 + a_1\omega_n^2)/2\omega_n$$

$$\zeta_n = (a_0/2\omega_n) + (a_1\omega_n/2)$$

Now we choose damping ratios ζ_i and ζ_j for two modes (natural frequencies ω_i and ω_j) and solve for the coefficients a_0 and a_1 (two equations in two unknowns).

Variation of Classical (Rayleigh) Damping with Frequency

Damping defined by $\zeta = (a_0/2\omega)+(a_1\omega/2)$ results in the variation shown by the combined curve below, which has the desirable feature of being somewhat uniform within a frequency range of interest (say 1 Hz to 7 Hz or 2π to 14π in radians/s).



Notes

1) For a choice of $\zeta_i = \zeta_j = \zeta$ ← same damping ratio in the two modes, we get

,

$$a_0 = \zeta \frac{2\omega_i\omega_j}{\omega_i + \omega_j}$$

$$a_1 = \zeta \frac{2}{\omega_i + \omega_j}$$

2) Classical damping and is attractive because of combination of mass and stiffness, allowing the no-damping free-vibration mode shapes to un-couple the matrix equation of motion.

Caughey damping

The above procedure was generalized by Caughey to allow for more control over damping in the specified modes of interest (i.e. to be able to specify ζ for more than 2 modes i and j)

In this generalization, you can stay within the scope of classical damping by using

$$\mathbf{c} = \mathbf{m}\sum_{i=0}^{N-1} a_i \left[\mathbf{m}^{-1}\mathbf{k}\right]^i$$

to find a_i coefficients to match ζ_i modal damping ratios, see for instance "Dynamics of Structures" by A. Chopra.

Disadvantages:

1. **c** can become a full matrix instead of being a banded matrix (if **m** and **k** are banded) as with $\mathbf{c} = a_0\mathbf{m} + a_1\mathbf{k}$

2. You must check to ensure that you don't end up with a negative ζi in some mode where you have not specifically specified damping (because damping variation with frequency might display sharp oscillations).

In summary, $\mathbf{c} = a_0\mathbf{m} + a_1\mathbf{k}$ is the usual choice at present despite the limitations discussed above.