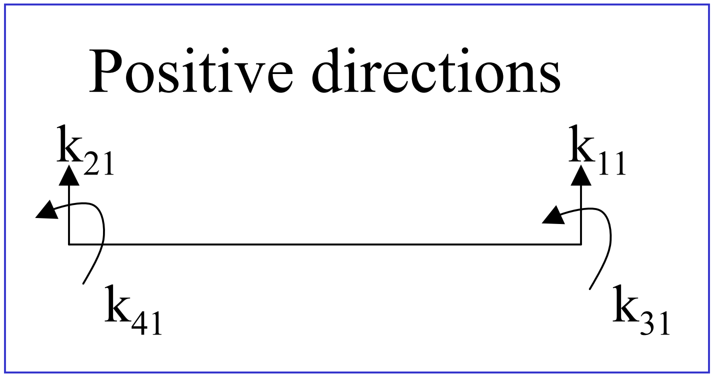
Stiffness Coefficients for a Flexural Element

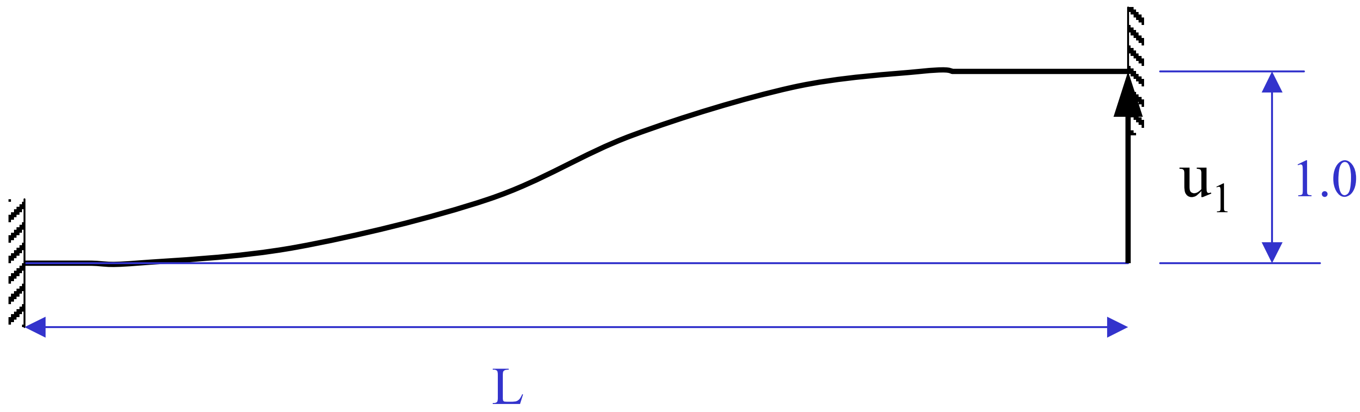
Ahmed Elgamal

Stiffness coefficients for a flexural element (neglecting axial deformations), Appendix 1, Ch. 1 Dynamics of Structures by Chopra.

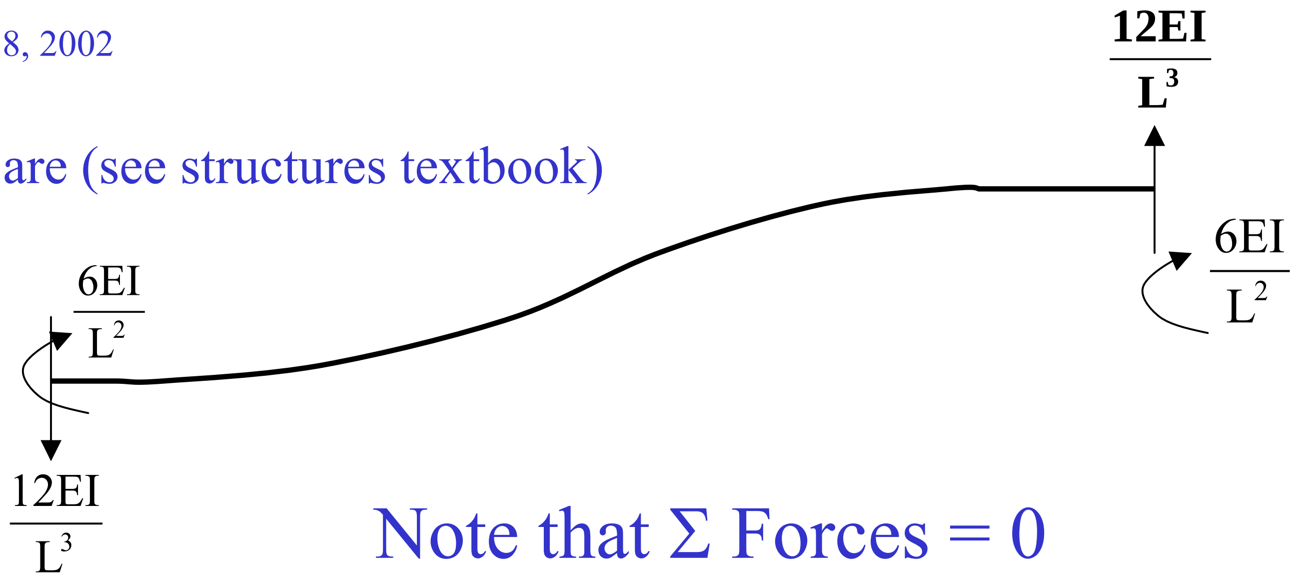
4 degrees of freedom



To obtain k coefficients in 1^{st} column of stiffness matrix, move $u_1 = 1$, $u_2 = u_3 = u_4 = 0$, and find forces and moments needed to maintain this shape.



These are (see structures textbook)

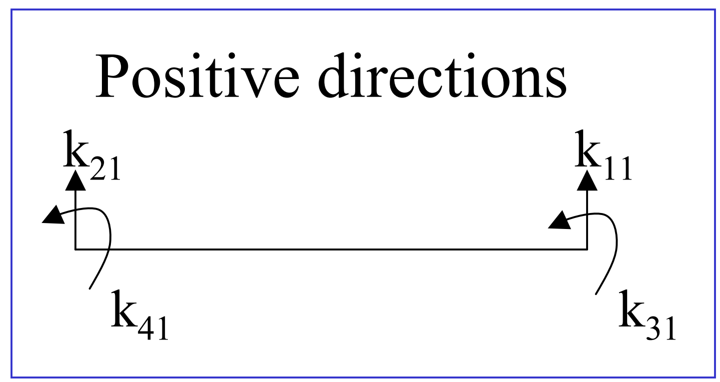


Note that Σ Forces = 0

Σ Moments = 0

$$\Sigma M = \frac{12EI}{L^3} - \frac{12EI}{L^3} = 0$$

i.e. remember $\mathbf{\frac{12EI}{L^3}}$, and you can find other forces & moments

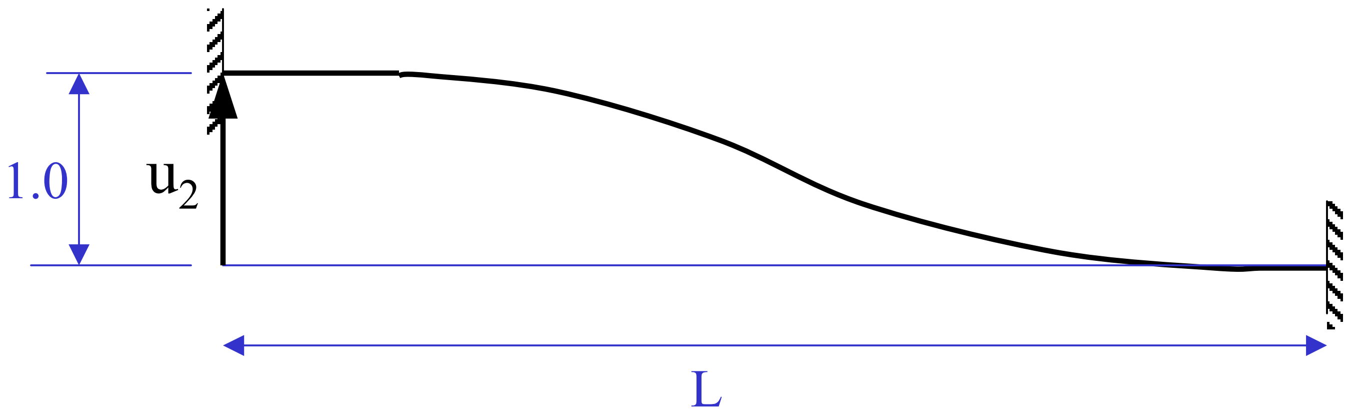


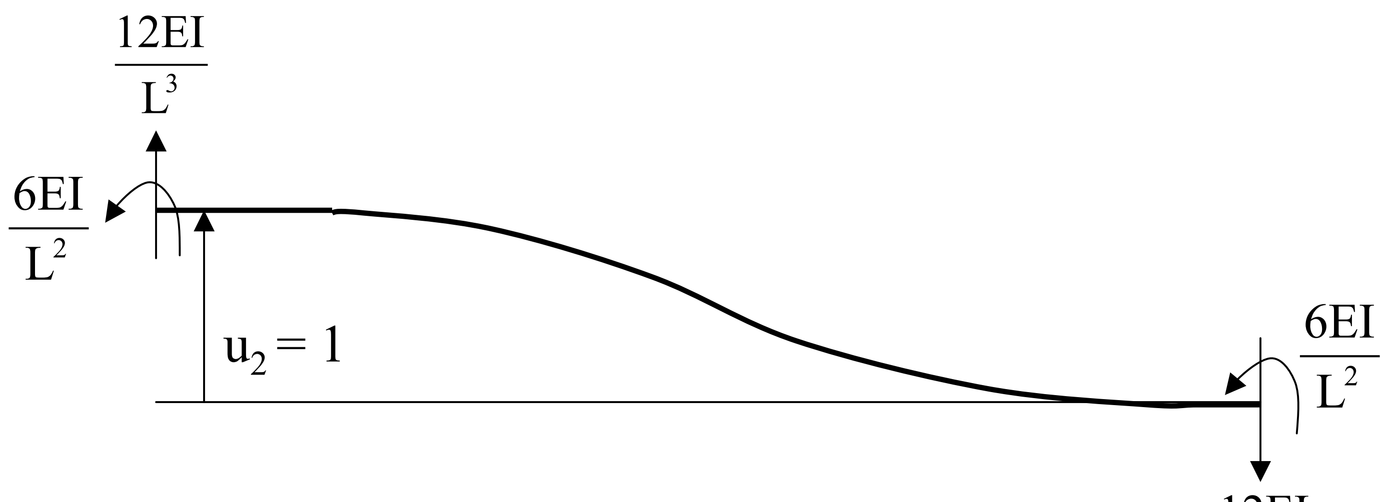
$$\underline{k} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

$k_{ij} = \underline{k}$, where i is row number and j is column number

$$\underline{k} = \frac{EI}{L^3} \begin{bmatrix} 12 & & & \\ -12 & & & \\ -6L & & & \\ -6L & & & \end{bmatrix}$$

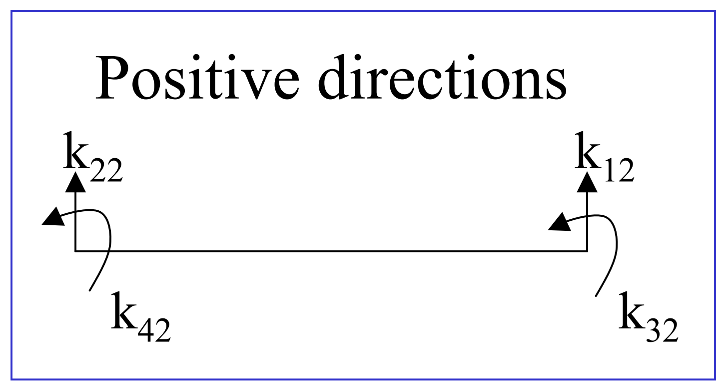
$$u_2 = 1, u_1 = u_3 = u_4 = 0$$



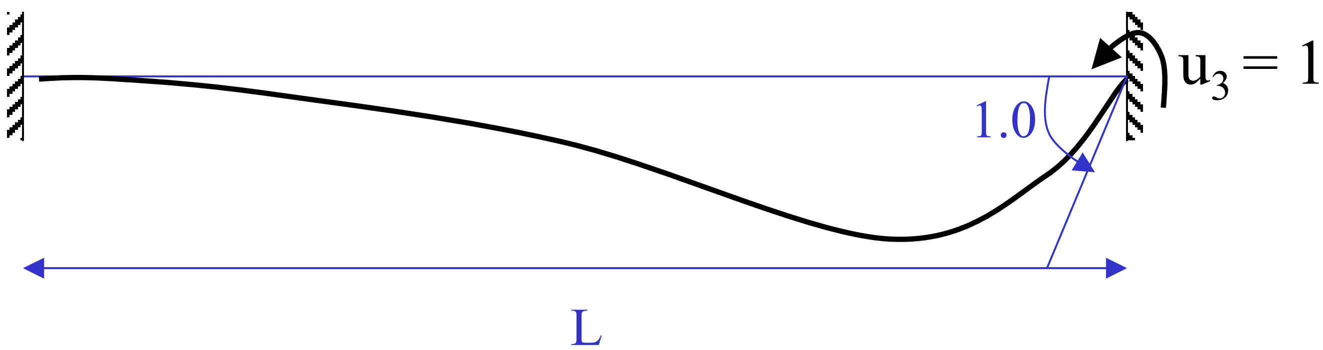


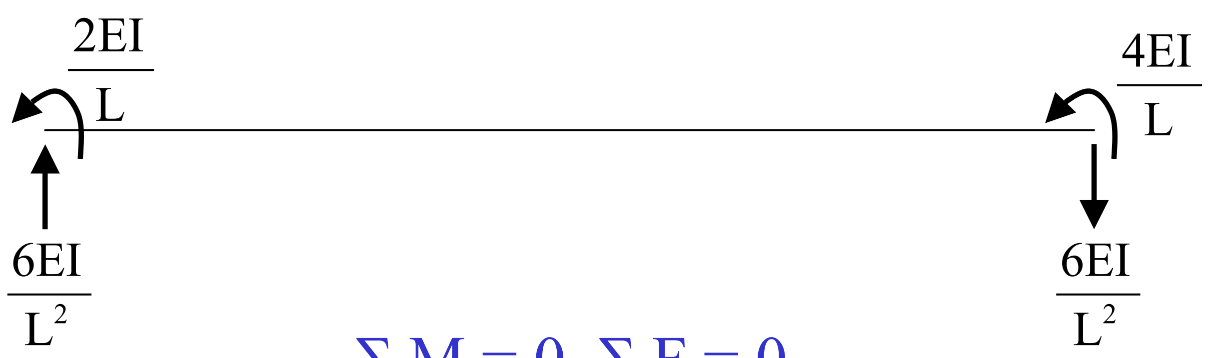
$$\Sigma M = 0, \Sigma F = 0$$

$$\underline{k} = \frac{EI}{L^3}\begin{bmatrix} -12 \\ 12 \\ 6L \\ 6L \end{bmatrix}$$



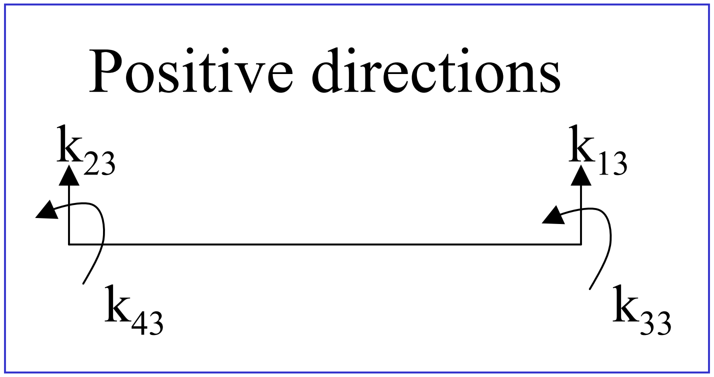
$$u_3 = 1,\ u_1 = u_2 = u_4 = 0$$



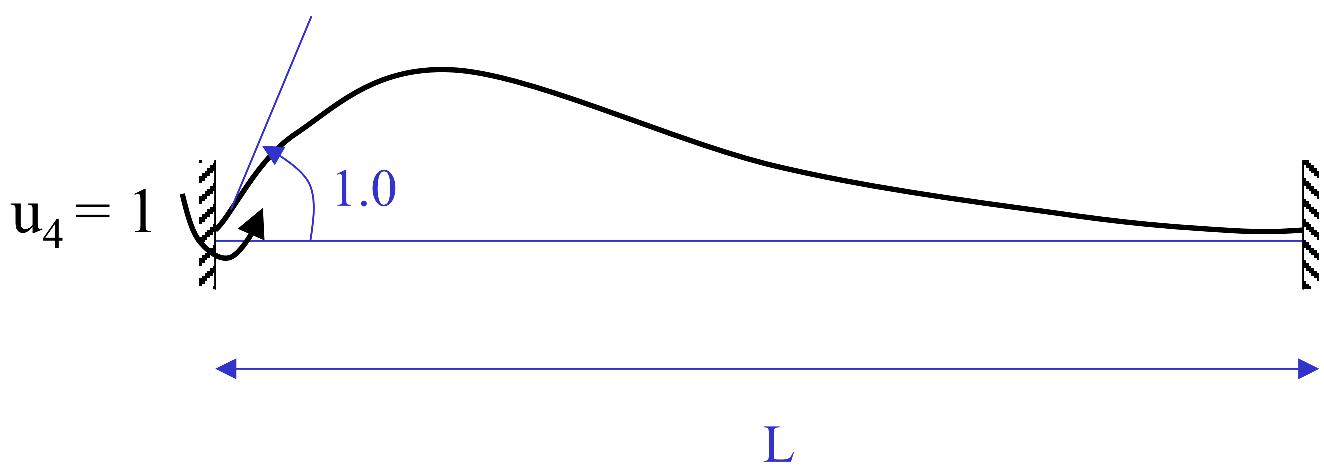


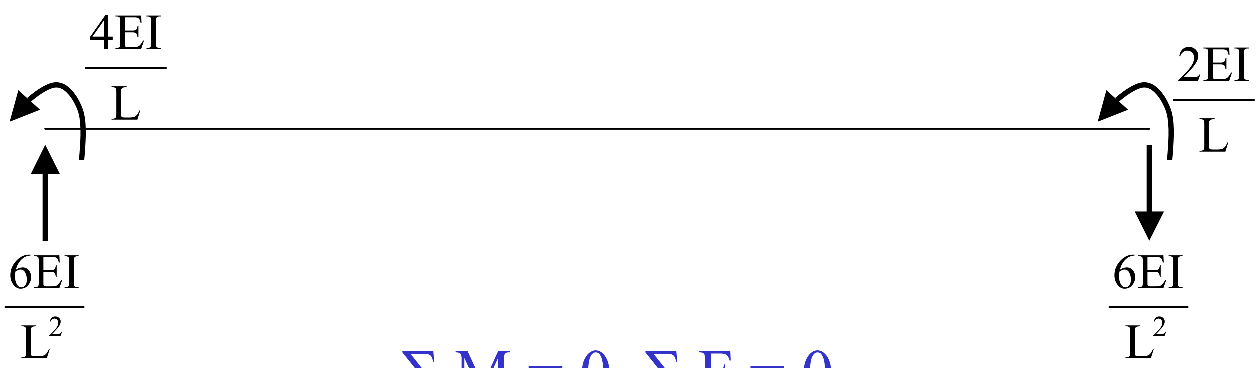
$$\Sigma M = 0, \Sigma F = 0$$

$$\underline{k} = \begin{bmatrix} \frac{-6EI}{L^2} \\ \frac{6EI}{L^2} \\ \frac{4EI}{L} \\ \frac{2EI}{L} \end{bmatrix}$$



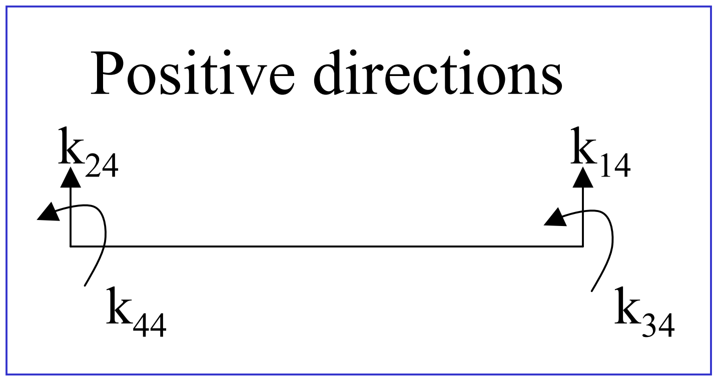
$u_4 = 1, u_1 = u_2 = u_3 = 0$



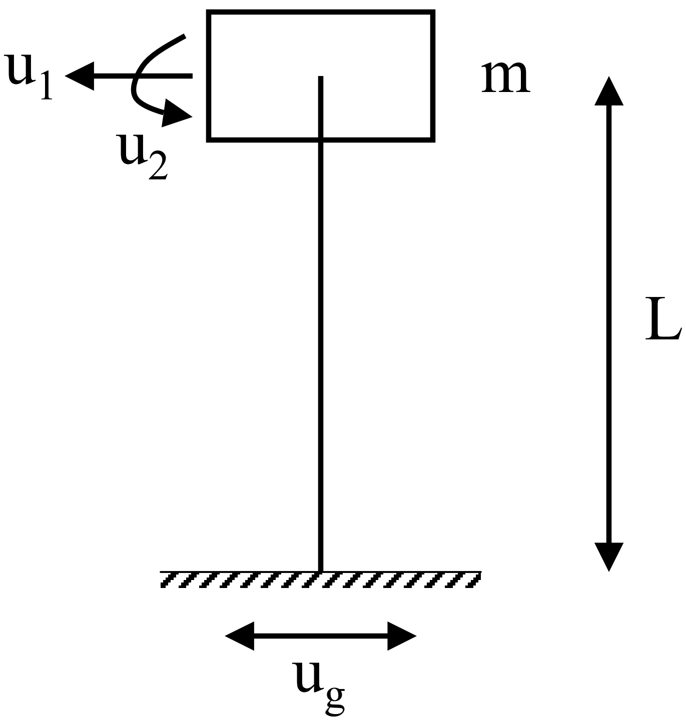


$$\Sigma M = 0, \Sigma F = 0$$

$$\underline{k} = \begin{bmatrix} & & & \frac{-6EI}{L^2} \\ & & & \frac{6EI}{L^2} \\ & & & \frac{2EI}{L} \\ & & & \frac{4EI}{L} \end{bmatrix}$$



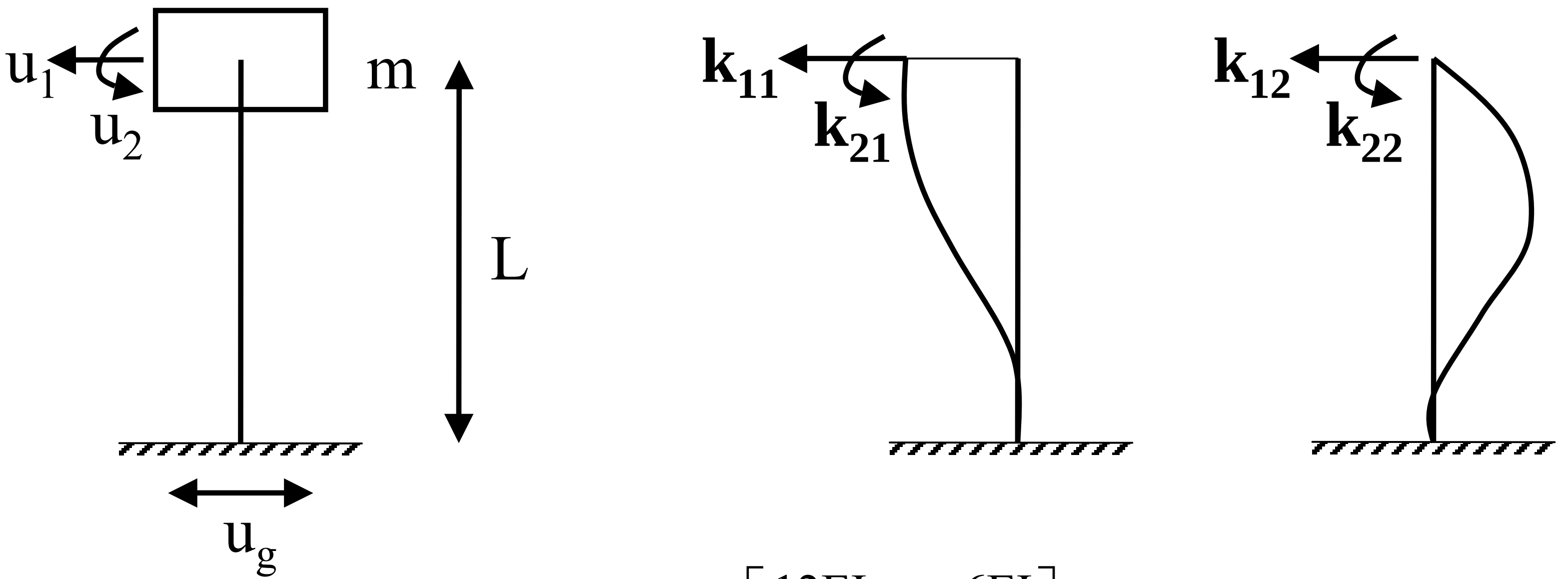
Example: Water Tank



m is lumped at a point & does not contribute in rotation

u_2 above was u_3 in the earlier section of these notes

Example: Water Tank (continued)



$$\begin{bmatrix} m & \\ & 0 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} m \\ 0 \end{bmatrix}\ddot{u}_g$$

"Note Symmetry"

Rotational (used to be u_3)

Example: Water Tank (continued)

Static Condensation:

Way to solve a smaller system of equations by eliminating degrees of freedom with zero mass.

e.g., in the above, the 2nd equation gives

$$\frac{-6EI}{L^2}u_1 + \frac{4EI}{L}u_2 = 0$$

or

$$u_2 = \frac{6EI}{L^2}\frac{L}{4EI}u_1 = \frac{6}{4L}u_1 = \frac{3}{2L}u_1 \quad \text{-----}*$$

Example: Water Tank (continued)

Substitute * into Equation 1

$$m\ddot{u}_1 + \left(\frac{12EI}{L^3} - \frac{6EI}{L^2}\frac{3}{2L}\right)u_1 = -m\ddot{u}_g$$

or,

$$m\ddot{u}_1 + \left(\frac{24EI - 18EI}{2L^3}\right)u_1 = -m\ddot{u}_g$$

or,

$$m\ddot{u}_1 + \left(\frac{3EI}{L^3}\right)u_1 = -m\ddot{u}_g$$

Now, solve for u_1 and u_2 can be evaluated from Equation * above.

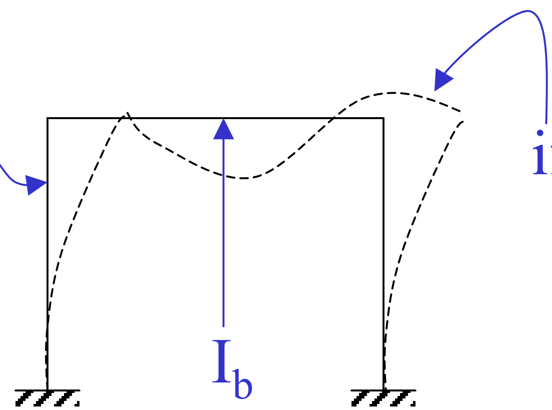
Static condensation can be applied to large MDOF systems of equations, the same way as shown above.

Example: Water Tank (continued)

$$m\ddot{u}_1 + \left(\frac{3EI}{L^3}\right)u_1 = -m\ddot{u}_g$$

k of water tank as we were given earlier.

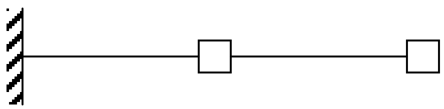
or of column



if $EI_b = 0$

Mandatory Reading

Example 9.4 page 362-364



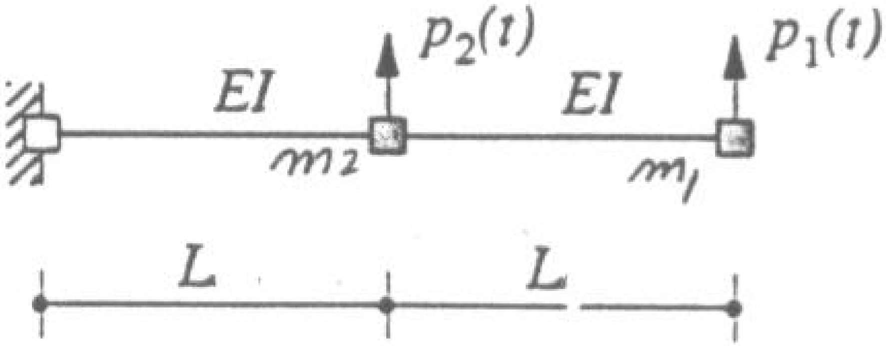
bending beam

Example 9.8 page 368-369

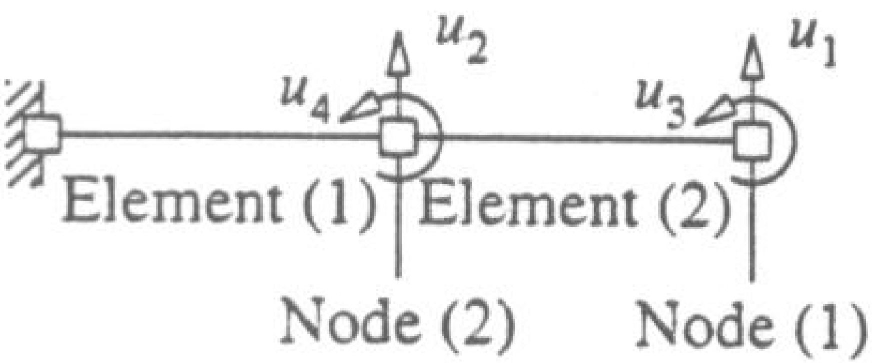
Sample Exercises: 9.5, 9.8, & 9.9

Example

(a)

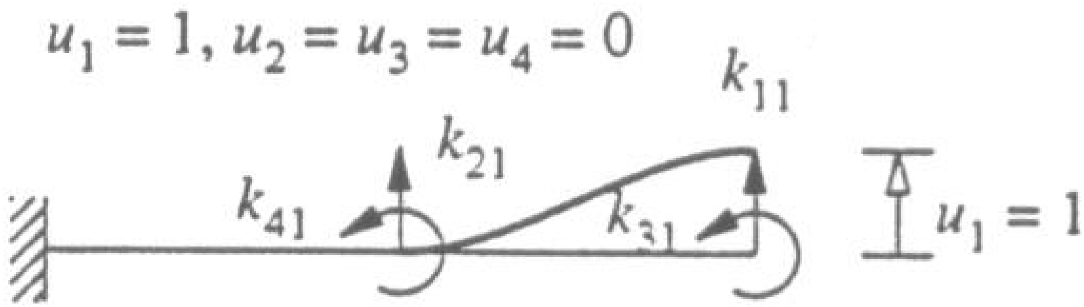


(b)

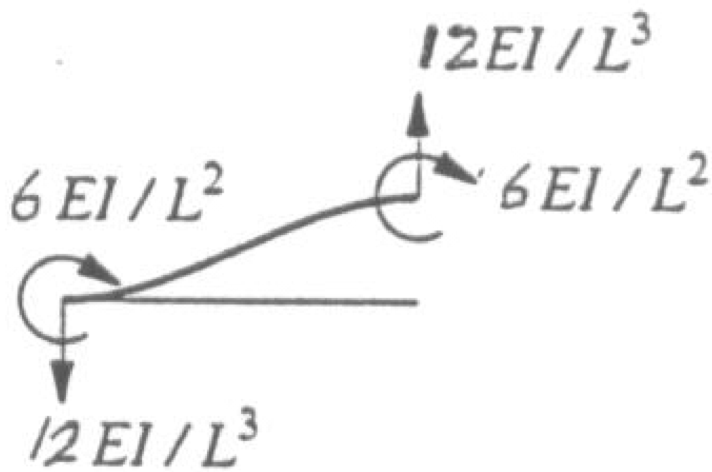


Reference: Dynamics of Structures, Anil K. Chopra, Prentice Hall, New Jersey, ISBN 0-13-086973-2

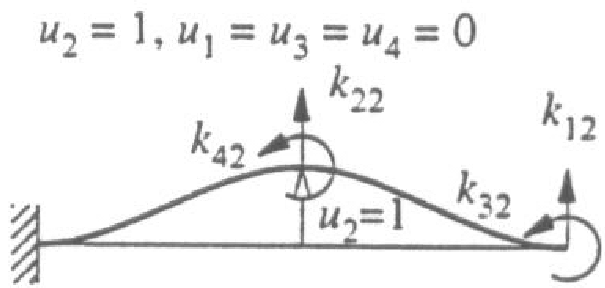
(c)



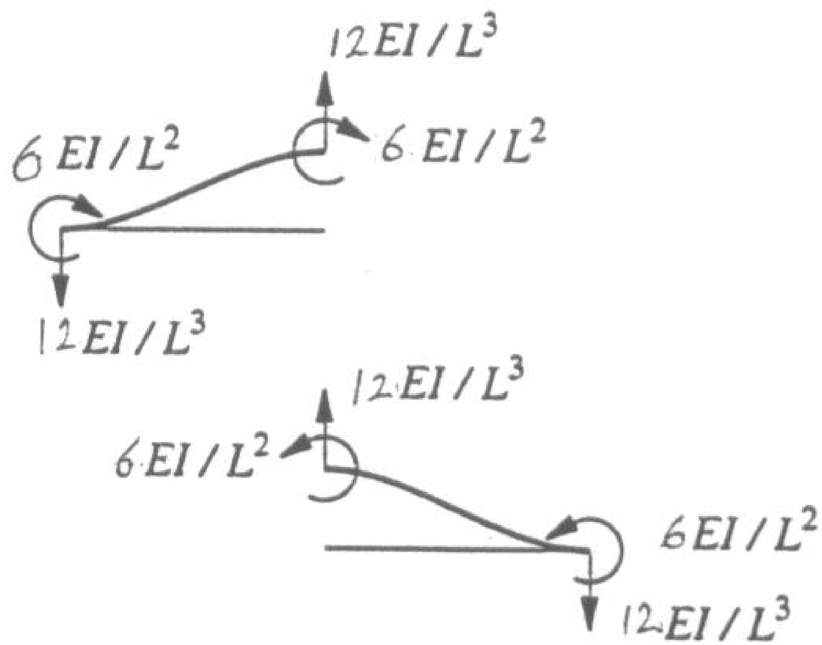
(d)



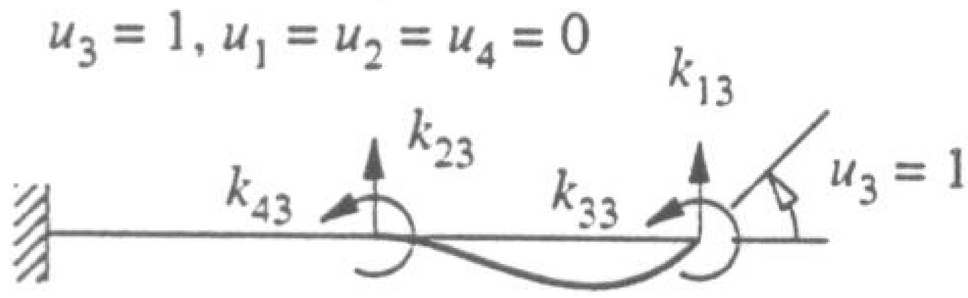
(e)


$u_2 = 1, u_1 = u_3 = u_4 = 0$
k_{22}
k_{12}
k_{42}
k_{32}
$u_2=1$

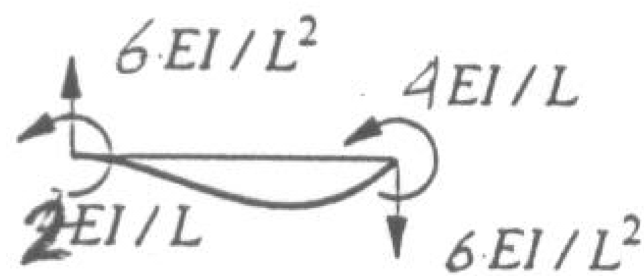
(f)


$12EI/L^3$
$6EI/L^2$
$6EI/L^2$
$12EI/L^3$
$12EI/L^3$
$6EI/L^2$
$6EI/L^2$
$12EI/L^3$

(g)

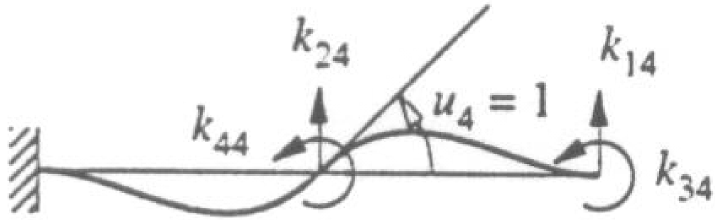


(h)

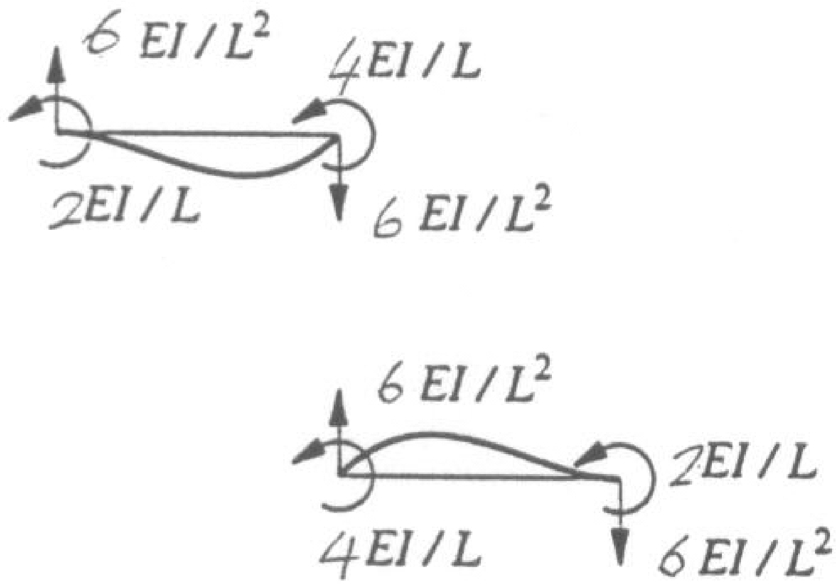


$u_4 = 1, u_1 = u_2 = u_3 = 0$

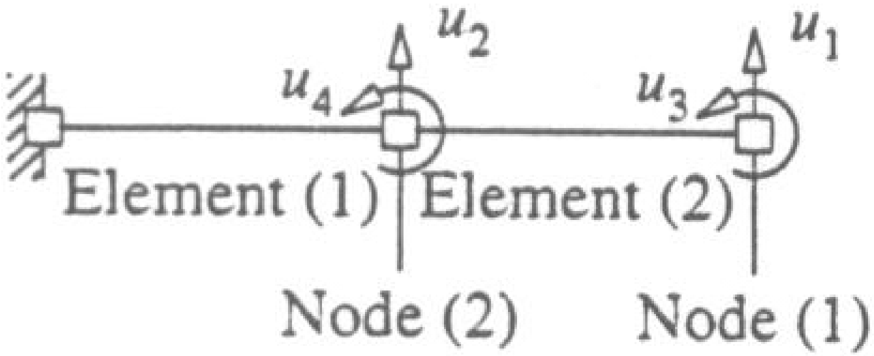
(i)



(j)



therefore,

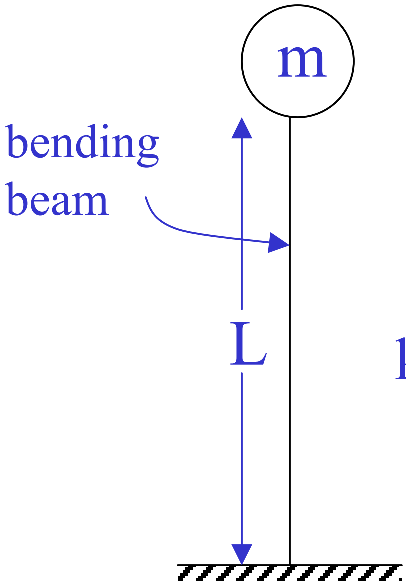


$$\underline{k} = \frac{EI}{L^3}\begin{bmatrix} 12 & -12 & -6L & -6L \\ -12 & 24 & 6L & 0 \\ -6L & 6L & 4L^2 & 2L^2 \\ -6L & 0 & 2L^2 & 8L^2 \end{bmatrix} \quad , \quad \underline{M} = \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

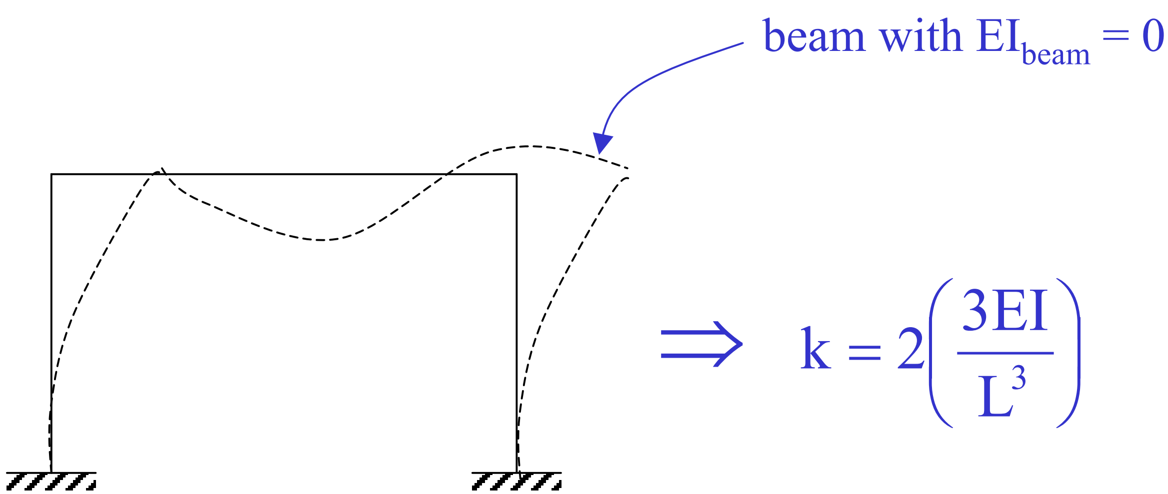
Sample Exercise: For the above cantilever system, write equation of motion and perform static condensation to obtain a 2 DOF system.

Reference: Dynamics of Structures, Anil K. Chopra, Prentice Hall, New Jersey, ISBN 0-13-086973-2

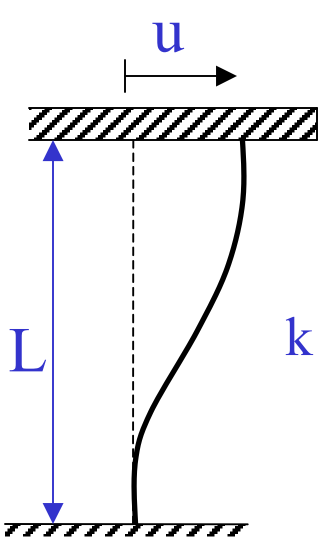
Column Stiffness (lateral vibration)



$$k = \frac{3EI}{L^3} \quad , \quad \omega = \sqrt{\frac{k}{m}}$$

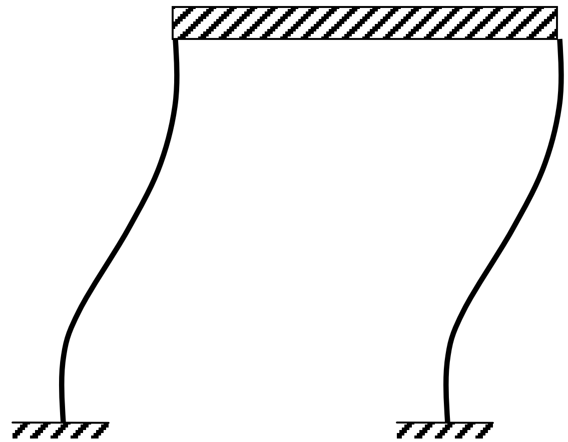


$$\Rightarrow \quad k = 2\left(\frac{3EI}{L^3}\right)$$

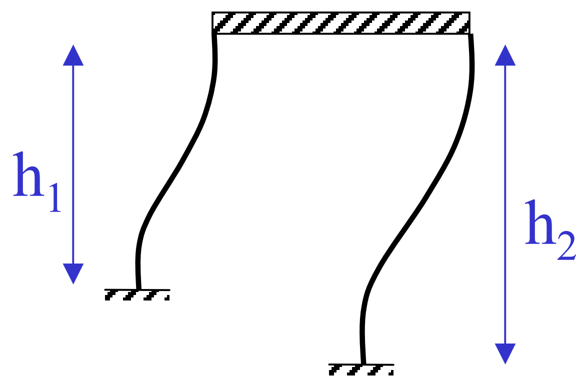


$$k = \frac{12EI}{L^3}$$

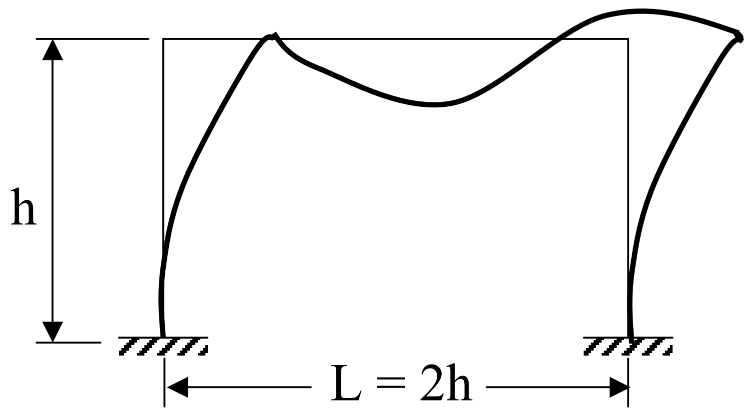
Case of $EI_b = \infty$ (rigid roof)



$$k = 2\left(\frac{12EI}{L^3}\right)$$



$$k = \frac{12E_1I_1}{h_1^3} + \frac{12E_2I_2}{h_2^3}$$

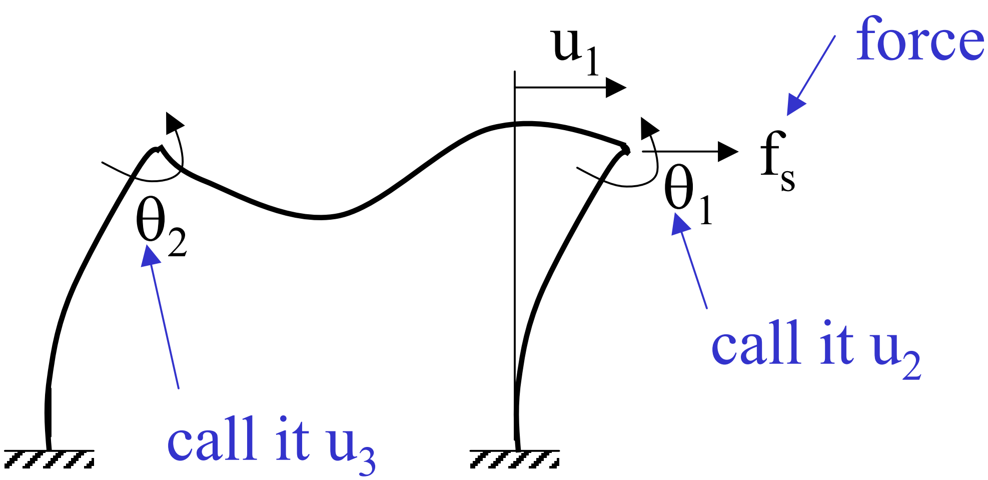


(See example 1.1 in Dynamics of Structures by Chopra)

$$k = \frac{96EI_c}{7h^3} \quad \text{if} \quad EI_b = EI_c$$

$$k = \frac{24EI_c}{h^3}\frac{12\rho+1}{12\rho+4}, \qquad \rho = \frac{I_b}{4I_c} \;\&\; E_c = E_b = E$$

Obtained by "static condensation" of 3x3 system



$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}f_s\\0\\0\end{bmatrix}$$

Use to represent u_2 and u_3 in terms of u_1 & plug back into

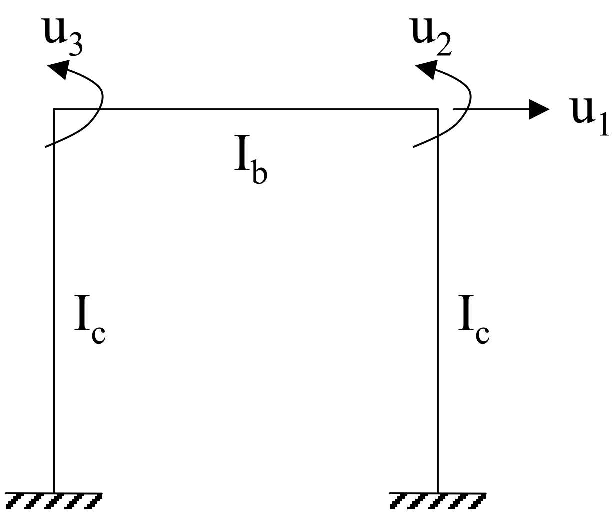
Technique can also be used for large systems of equations

and get $f_s = ku_1$

(See example 1.1 in Dynamics of Structures by Chopra)

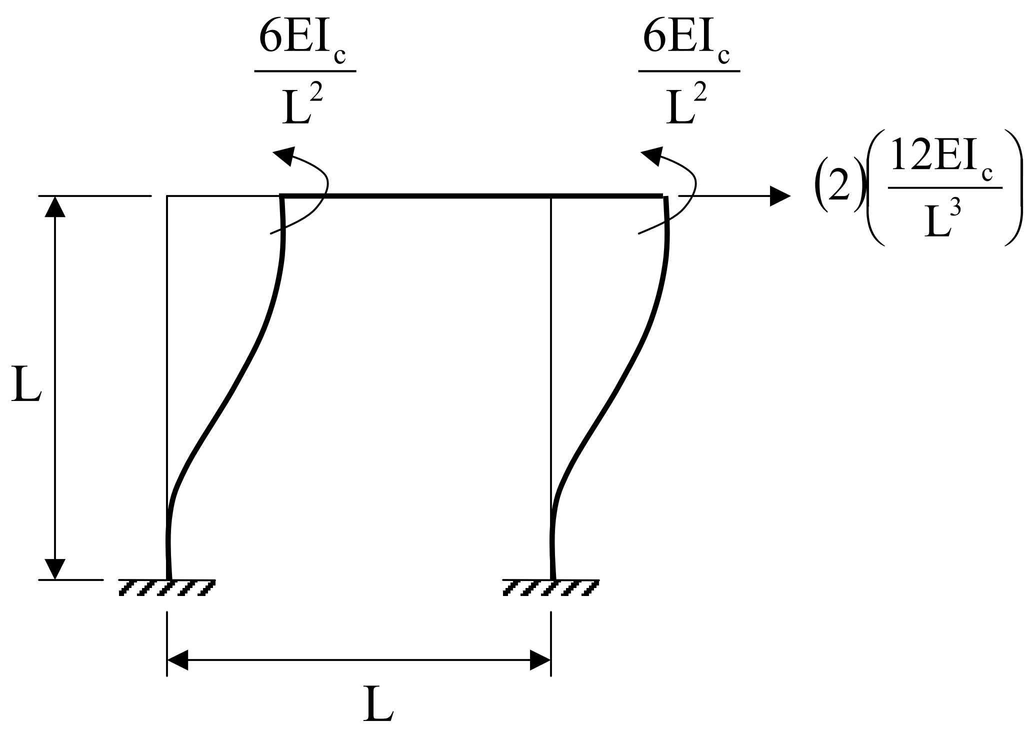
Draft Example

Neglect axial deformation



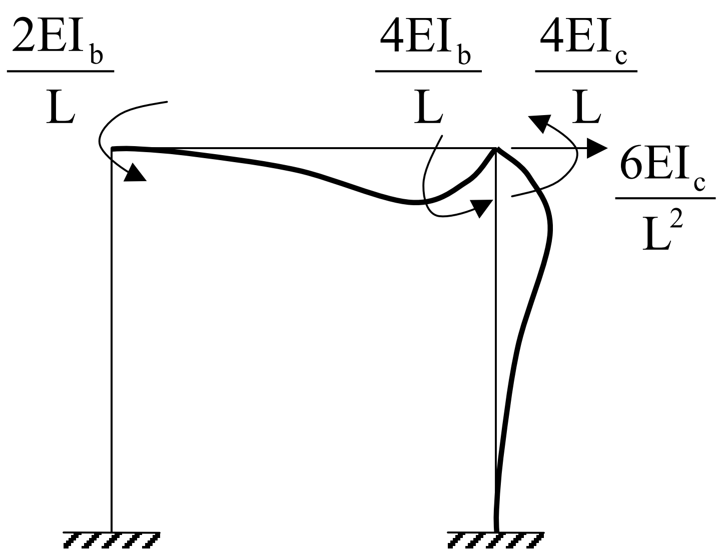
$$u_1 = 1$$

$$u_2 = u_3 = 0$$



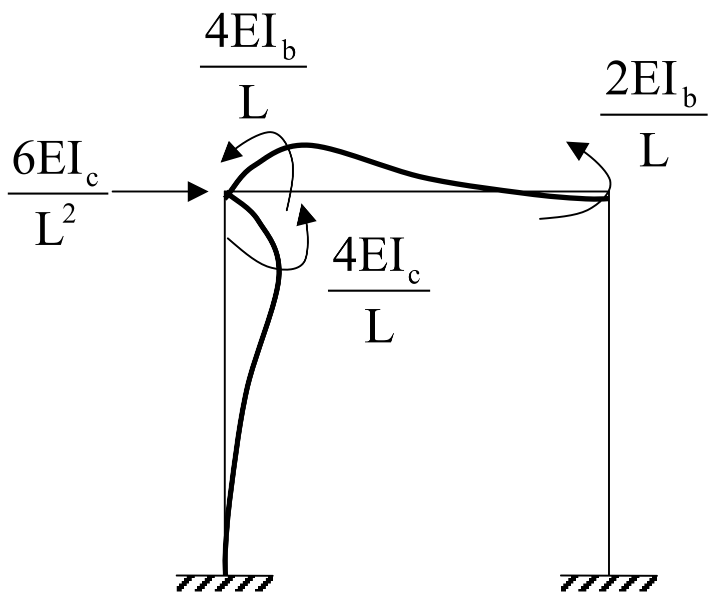
$$u_2 = 1$$

$$u_1 = u_3 = 0$$



$$u_3 = 1$$

$$u_1 = u_2 = 0$$



$$\underline{k} = \frac{E}{L^3}\begin{bmatrix} 24I_c & 6I_cL & 6I_cL \\ 6I_cL & 4(I_b + I_c)L^2 & 2I_bL^2 \\ 6I_cL & 2I_bL^2 & 4(I_b + I_c)L^2 \end{bmatrix}$$

If frame is subjected to lateral force f_s

Then (for simplicity, let $I_c = I_b = I$)

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_s \\ 0 \\ 0 \end{bmatrix}$$

Static condensation:

From 2nd and 3rd equations,

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = -\begin{bmatrix} 8L^2 & 2L^2 \\ 2L^2 & 8L^2 \end{bmatrix}^{-1} \begin{bmatrix} 6L \\ 6L \end{bmatrix} u_1$$

$$= \frac{-1}{64L^4 - 4L^4} \begin{bmatrix} 8L^2 & -2L^2 \\ -2L^2 & 8L^2 \end{bmatrix}^{-1} \begin{bmatrix} 6L \\ 6L \end{bmatrix} u_1$$

$$= \frac{-6}{10L} \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_1$$

Note matrix inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

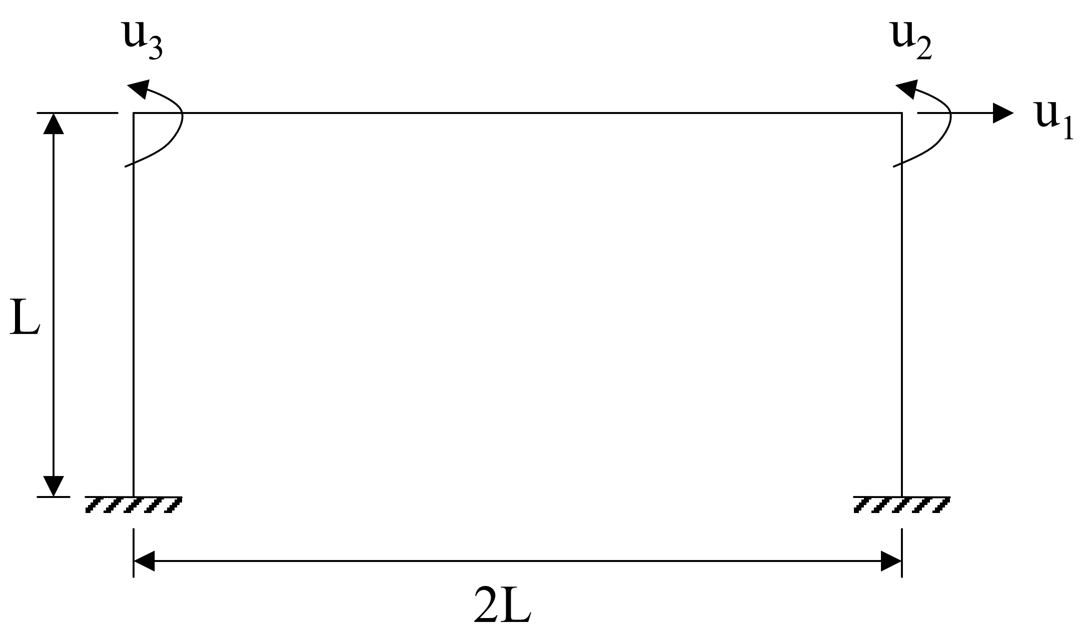
Substitute into 1st equation

$$\frac{EI}{L^3}\left[24-\frac{36}{10}-\frac{36}{10}\right]u_1 = f_s = \frac{168EI}{10L^3}u_1$$

or

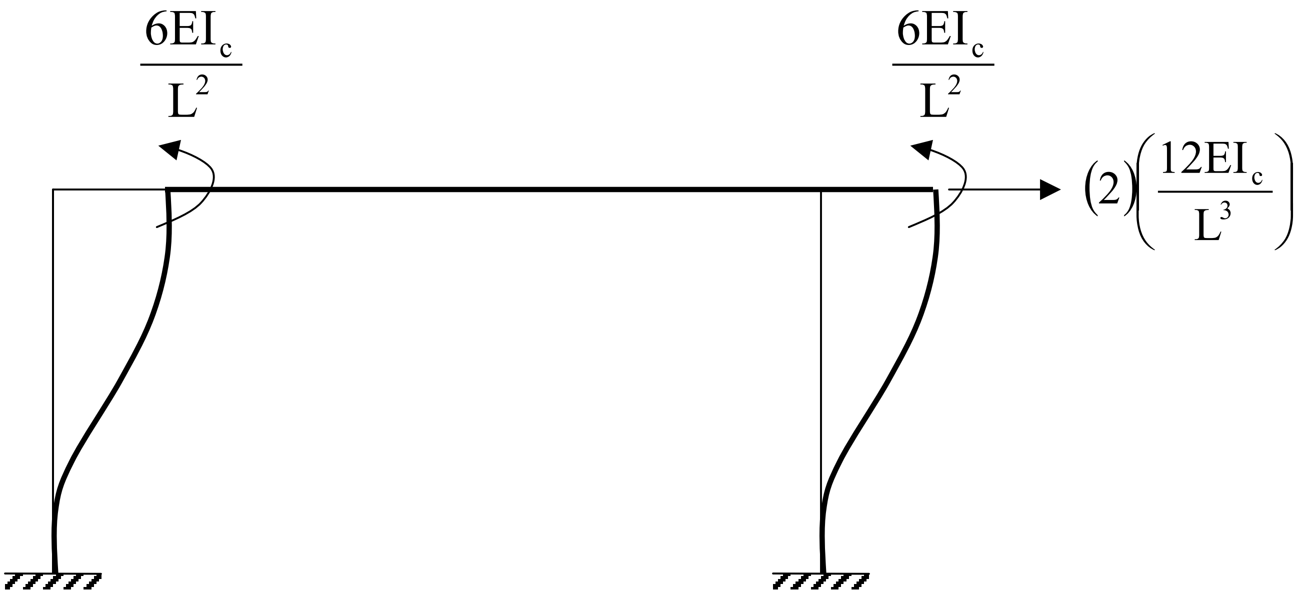
$$k = \frac{168EI}{10L^3}$$ (check this result)

Draft Example 2



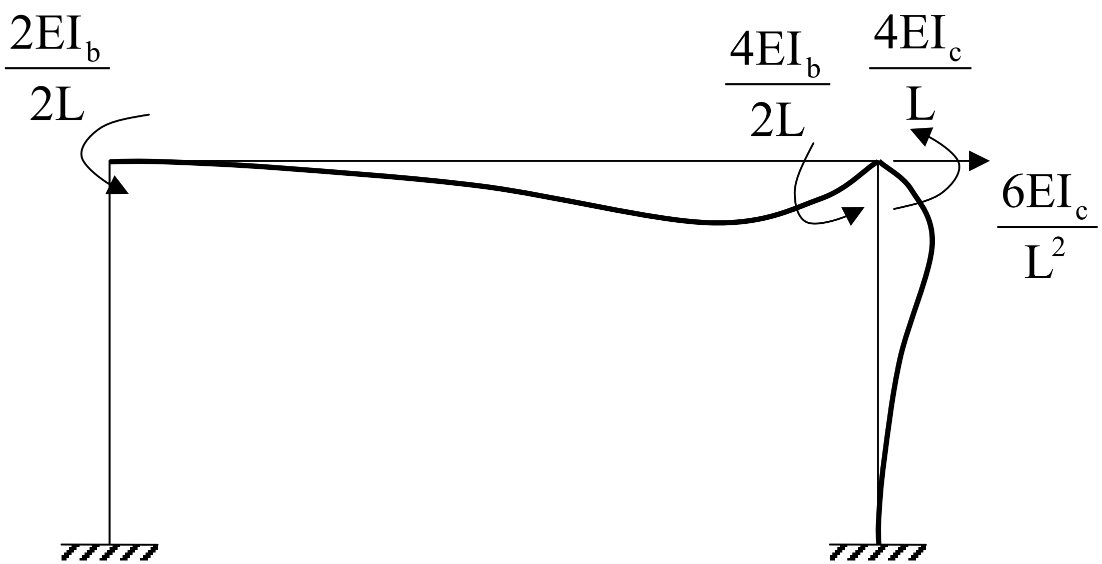
$$u_1 = 1$$

$$u_2 = u_3 = 0$$



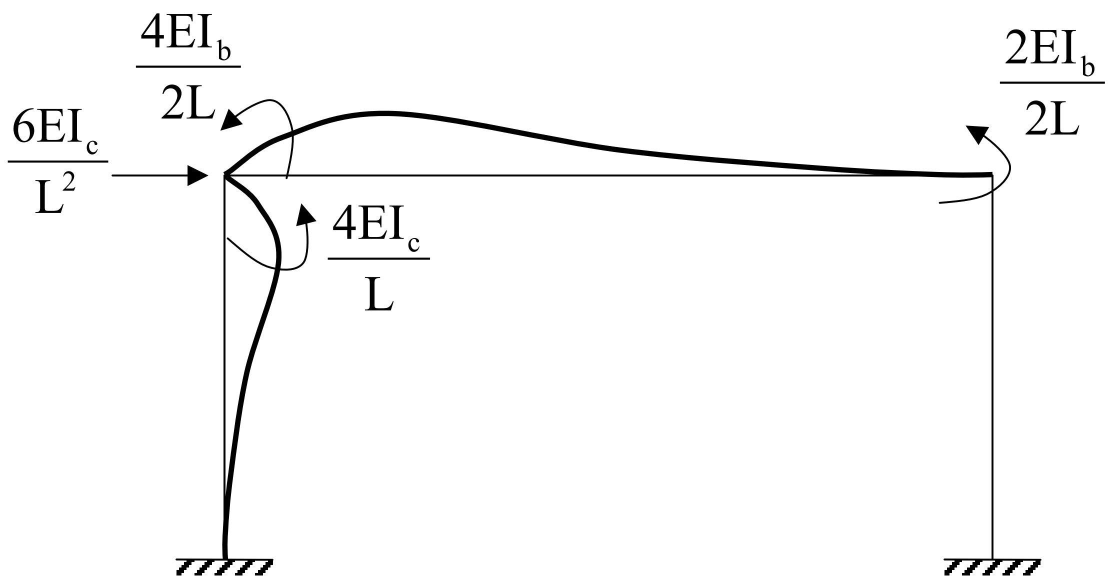
$$u_2 = 1$$

$$u_1 = u_3 = 0$$



$$u_3 = 1$$

$$u_1 = u_2 = 0$$



$$\underline{k} = \frac{E}{L^3}\begin{bmatrix} 24I_c & 6I_cL & 6I_cL \\ 6I_cL & 4\left(\frac{I_b}{2}+I_c\right)L^2 & I_bL^2 \\ 6I_cL & I_bL^2 & 4\left(\frac{I_b}{2}+I_c\right)L^2 \end{bmatrix}$$

For simplicity, let $I_b = I_c$

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 6L & 6L \\ 6L & 6L^2 & L^2 \\ 6L & L^2 & 6L^2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_s \\ 0 \\ 0 \end{bmatrix}$$

Static condensation:

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = -\begin{bmatrix} 6L^2 & L^2 \\ L^2 & 6L^2 \end{bmatrix}^{-1} \begin{bmatrix} 6L \\ 6L \end{bmatrix} u_1$$

$$= \frac{-1}{36L^4 - L^4} \begin{bmatrix} 6L^2 & -L^2 \\ -L^2 & 6L^2 \end{bmatrix}^{-1} \begin{bmatrix} 6L \\ 6L \end{bmatrix} u_1$$

$$= \frac{-30}{35L} \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_1 = \frac{-6}{7L} \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_1$$

Substitute in 1st Equation

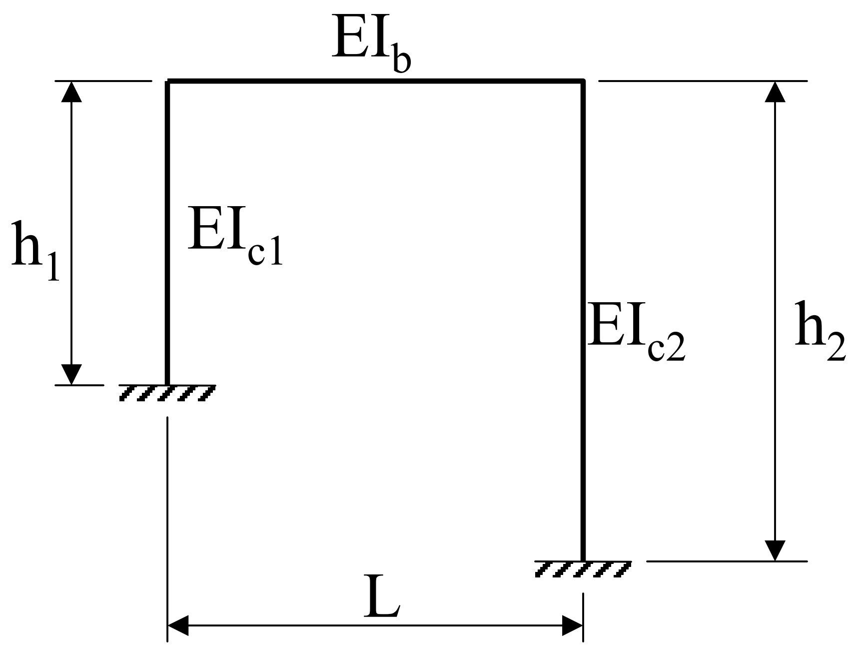
$$\frac{EI}{L^3}\left[24-\frac{36}{7}-\frac{36}{7}\right]u_1 = f_s$$

or, $$f_s = \frac{96}{7}\frac{EI}{L^3}u_1$$

or, $$k = \frac{96EI}{7L^3}$$ ← Same as in Example 1.1, Dynamics of Structures by Chopra

Sample Exercise

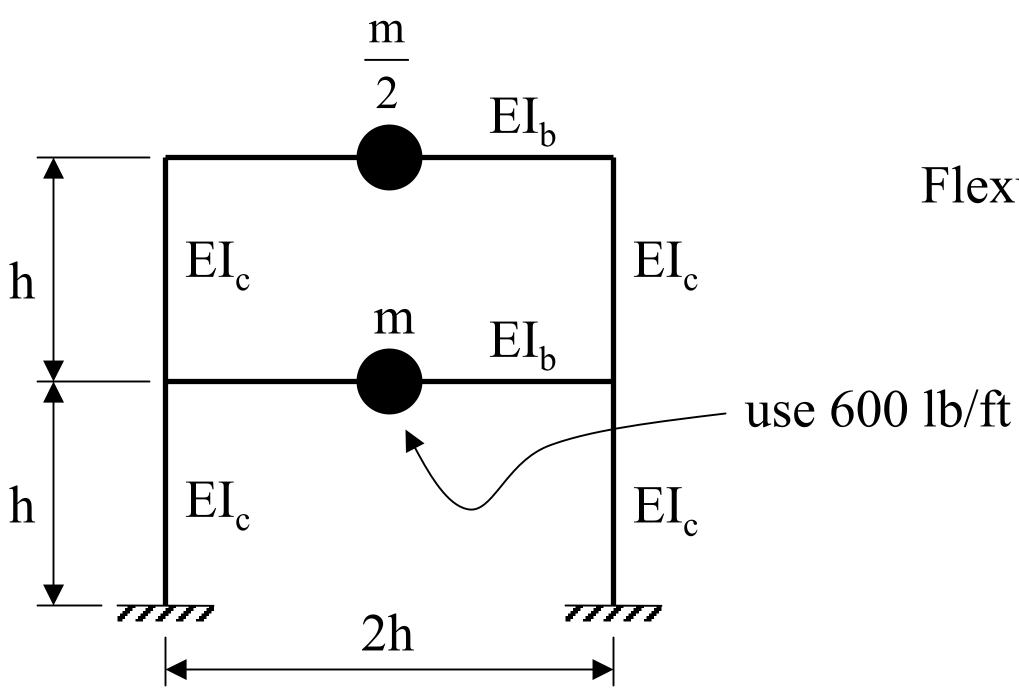
1) 1.1 Derive stiffness matrix $\underline{k}$ for



1.2 For the special case of $I_{c1} = I_{c2} = I_b$, $h_1 = h_2 = h$ and $L = 2h$, find lateral stiffness k of the frame.

Sample Exercise

2) Derive equation of motion for:



Flexural rigidity of beams and columns

$E = 29{,}000$ ksi,

Columns W8x24 sections

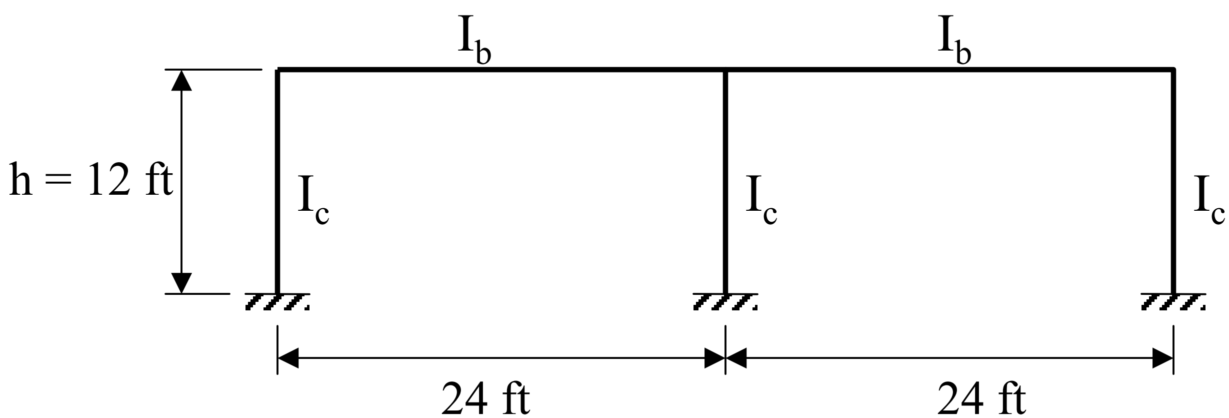
with $I_c = 82.4\ in^4$

$h = 12$ ft

$I_b = \frac{1}{2} I_c$

Sample Exercise (Optional)

3) Derive lateral k of system (need to use computer to invert 3x3 matrix)



E = 29,000 ksi,

$I_c = 82.4\ in^4$ ← W8x24 sections

$I_b = \frac{1}{2} I_c$